AN ACCURACY ANALYSIS OF AN AD HOC LOWER CONFIDENCE LIMIT PROCEDURE FOR SYSTEM AVAILABILITY

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THESIS

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ABSTRACT

The accuracy of proposed lower confidence limits for system availability is analyzed. Random values of the lower $100(1-\alpha)\%$ confidence limit $\hat{A}_{S\,L(\alpha)}$ for system availability are computed for a system whose failure density is exponential (λ) and whose repair density is exponential (μ). The system is modeled as an alternating renewal process. The $100(1-\alpha)$th percentile point of the generated distribution of $\hat{A}_{SL(\alpha)}$ is compared with system availability as a measure of accuracy for $\hat{A}_{S\,L(\alpha)}$.

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I. INTRODUCTION

The system considered in this thesis is of the type which operate for a time, fail, are repaired and returned to operation. The system then may be said to have two states and can be modeled as a renewal process.

The objectives of this paper were the following:

1. To compute exact values of the steady state availability for a system with given distributions of times to failure and times to repair.

2. To compare the availability of a system based on the test procedures used with the $100(1-\alpha)$ th percentile point of the distribution of a proposed $100(1-\alpha)\%$ lower confidence limit. This will allow an accuracy analysis of a proposed lower confidence limit.

The method used was to fit a normal distribution to simulated data and compute the $100(1-\alpha)$ th percentile point of the distribution.

An alternative method was investigated, which fitted a 2 parameter gamma distribution. The procedure proved to be inaccurate.

II. STATEMENT OF THE PROBLEM AND ANALYSIS

A. DEFINITION OF AVAILABILITY

In this paper the term availability [1] is defined as the probability that a system will be "up" or operable when called upon at a random time.

Normally, availability is not specified alone. Both reliability and maintainability are usually specified together with availability. For durable, continuous operated hardware, reliability can be specified by mean-time-to-failure (MTTF). Given the exponential form of time to failure distribution, which is applicable to most types of complex equipment, MTTF is a constant, the reciprocal of the failure rate.

Maintainability can be specified by mean-time-to-repair (MTTR) and mean-preventive-maintenance-time. Corrective maintenance times are often found to be well described by the exponential distribution. Given that the time-to-repair is distributed exponential, MTTR is a constant, the reciprocal of the repair rate. Accordingly we make the following definitions.

(1) $A(t)$ - Availability at time t, or the probability that an item will be operable at a stated instant in time. $t \geq 0$

(2) $\overline{A}_T$ - Interval availability is the time average of $A(t)$ during intervals of length T, and is readily obtained from the equation for the average value of a function.

$$\overline{A}_T = \frac{\int_0^T A(t)dt}{\int_0^T dt} = \frac{1}{T}\int_0^T A(t)\,dt$$

Alternately, it may be considered that there is some probability distribution h(t) on demand time. Then the interval availability is given by

$$E\,[A(t)] = \int_0^\infty A(t)\,h(t)\,dt$$

In the special case where h(t) is uniform on the interval [0, T]

$$h(t) = \begin{cases} \frac{1}{T} & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$

then $\overline{A}_T = \frac{1}{T}\int_0^T A(t)\,dt.$

(3) A - the steady-state availability is the limiting interval availability as $T \rightarrow \infty$

$$A = \lim_{T\rightarrow\infty} \frac{1}{T}\int_0^T A(t)\,dt$$

It can also be shown that

$$A = \lim_{T\rightarrow\infty} A(t)$$

If the components do not undergo checkout or repair its interval availability approaches zero. For systems with exponentially distributed failure times where

$$A(t) = e^{-\lambda t}$$

$$\overline{A}_T = \frac{1 - e^{-\lambda T}}{\lambda T}$$

$$\overline{A}_T \rightarrow A \rightarrow 0 \quad \text{as } T \rightarrow \infty$$

Thus the steady state or long term uptime ratio of a component which does not undergo repair approaches zero.

If the component is subject to repair, its steady state availability is not zero. Assume that both failure and repair times are exponentially distributed with means $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively. Given that A(0) = 1 i.e., component available at start of mission, then

$$A(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-t(\mu + \lambda)}$$

μ = repair rate

λ = failure rate

then $$A = \lim_{t \to \infty} A(t) = \frac{\mu}{\mu + \lambda}$$

$$= \frac{MTTF}{MTTF + MTTR}$$

Catron [2] has shown that exact values of A(t) can be and were computed for repair distributions other than exponential. The use of interval availability as $t \to \infty$ i.e.,

$$\overline{A}_T = \frac{1}{T} \int_0^T A(t)\, dt \to A$$

to approximate the true availability over the first mission time as is currently being done in practice, is a conservative procedure.

The amount by which the limiting interval availability underestimates the true availability during the first mission time is dependent upon the limiting value, i.e., the lower limiting value, the more it underestimates the true availability.

(4) A_S - System availability. For a series system the system availability is defined as:

$$A_S = \prod_{i=1}^{K} A_i$$

where A_i is the availability of the component of type i. $i = 1, 2, \cdots K$

B. PROBLEM

The problem considered in this paper is to check the accuracy of a proposed lower confidence limit procedure $\hat{A}_{S\,L(\alpha)}$ for A_S based upon component failure time data and repair time data.

(1) When assuming exponential failure rate α_i and exponential repair rate β_i then A_i is given by

$$A_i = \frac{\beta_i}{\beta_i + \alpha_i}$$

The following ranges of the failure rate (α_i) and repair rate (β_i) has been used in the simulation.

$$0.005 \leq \alpha_i \leq 0.01$$

$$3.0 \leq \beta_i \leq 10.0$$

which implies that the ranges of MTTF of component i, denoted by λ_i and MTTR of component i, denoted by μ_i are as follows.

$$100 \leq \lambda_i \leq 200$$

$$\frac{1}{10} \leq \mu_i \leq \frac{1}{3}$$

(2) The equation of component availability can be written as

$$A_i = \frac{\beta_i}{\beta_i + \alpha_i} = \frac{1}{1 + \frac{\alpha_i}{\beta_i}} = \frac{1}{1 + \alpha_i \mu_i}$$

but $A_i = \dfrac{1}{1 + \alpha_i \mu_i} = 1 - \alpha_i \mu_i + (\alpha_i \mu_i)^2 - (\alpha_i \mu_i)^3 + \cdots$

provided the series converges, which it does since $\alpha_i \mu_i < 1$.

Since the product $\alpha_i \mu_i$ is very small, in this case

$$0.0005 \leq \alpha_i \mu_i \leq 0.00333\ldots$$

$$A_i = \frac{1}{1+\alpha_i \mu_i} \doteq 1 - \alpha_i \mu_i$$

Then $A_S = \prod_{i=1}^{K} A_i = \prod_{i=1}^{K} (1-\alpha_i \mu_i) \doteq 1 - \sum_{i=1}^{K} \alpha_i \mu_i$

Since α_i and μ_i will have small variance we shall apply the central limit theorem, and fit a normal distribution to

$$A_S = 1 - \sum_{i=1}^{K} \alpha_i \mu_i$$

Then $\hat{A}_S = 1 - \sum_{i=1}^{K} \hat{\alpha}_i \hat{\mu}_i$

Assuming that $\hat{A}_S \sim N\left[A_S, \sum_{i=1}^{K} \left(\frac{n_i+1}{n_i} \frac{\alpha_i \mu_i^2}{\sum_{j=1}^{N_i} T_{oij}} \right)\right]$

implies that $\hat{A}_{SL(\alpha)} = \hat{A}_S - \left(\sum_{i=1}^{K} \frac{\hat{\alpha}_i \hat{\mu}_i^2}{\sum_{j=1}^{N_i} T_{oij}} \right)^{\frac{1}{2}} Z_\alpha$

C. ALTERNATIVE METHOD

An alternative method was investigated, which fitted a 2 parameter gamma distribution by method of moments to the distribution of

$$\hat{A}_S = \prod_{i=1}^{K} \hat{A}_i = \prod_{i=1}^{K} \frac{\hat{\beta}_i}{\hat{\beta}_i + \hat{\alpha}_i}$$

where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the estimates of α_i and β_i respectively.

Under the assumption that time to failure and time to repair of the components were distributed exponentially with failure rate α_i and repair rate β_i respectively. The following approach was used to make the fit:

$$-\ln A_S = \sum_{i=1}^{K} -\ln\left(\frac{\beta_i}{\beta_i+\alpha_i}\right) = \sum_{i=1}^{K} -\ln\left(1-\frac{\alpha_i}{\alpha_i+\beta_i}\right)$$

$$Q_i = 1-A_i = 1 - \frac{\alpha_i}{\alpha_i+\beta_i}$$

Thus by Taylor's expansion

$$-\ln A_S = \sum_{i=1}^{K} -\ln(1-Q_i) \doteq \sum_{i=1}^{K} \left(Q_i + \frac{Q_i^2}{2}\right)$$

$$\sum_{i=1}^{K} \left(Q_i + \frac{Q_i^2}{2}\right) \equiv \sum_{i=1}^{K} Y_i \equiv Y_s$$

$$\hat{Y}_s = \sum_{i=1}^{K} \hat{Y}_i = \sum_{i=1}^{K} \left(a_i \hat{Q}_i + b_i \frac{\hat{Q}_i^2}{2}\right)$$

a_i and b_i are choosen so that

$$E(\hat{Y}_i) = Y_i$$

and

$$\hat{Q}_i = 1 - \hat{A}_i = 1 - \frac{\hat{\alpha}_i}{\hat{\alpha}_i + \hat{\beta}_i}$$

where $\hat{\alpha}_i$ and $\hat{\beta}_i$ are the estimate of α_i and β_i respectively.

The 2 parameter gamma, $\Gamma(r, \gamma)$ was fitted by method of moments to the distribution of $\hat{Y}_s$.

$$E(\hat{Y}_s) = \frac{r}{\gamma}$$

$$Var(\hat{Y}_s) = \frac{r}{\gamma^2}$$

$$\frac{E(\hat{Y}_s)}{Var(\hat{Y}_s)} = \gamma$$

Therefore

$$\frac{E^2(\hat{Y}_s)}{Var(\hat{Y}_s)} = r$$

Then $$\hat{r} = \frac{\widehat{E^2(\hat{Y}_s)}}{\widehat{Var(\hat{Y}_s)}}$$

by replacing all α_i and β_i by $\hat{\alpha}_i$ and $\hat{\beta}_i$.

Then $2\gamma\hat{Y}_s$ is distributed χ^2_{2r}

$$-\ln A_S = Y_s = \frac{r}{\gamma}$$

$$A_S = e^{-\frac{r}{\gamma}}$$

From this assumption a one sided lower $100(1-\alpha)\%$ C. L. for A_S was computed.

i.e.,
$$P(2\gamma\hat{Y}_s \geq \chi^2_{\alpha,2r}) = 1-\alpha$$

$$P\left(e^{-\frac{r}{\gamma}} \geq \exp\left(\frac{-\hat{Y}_s\,[2\hat{r}]}{\chi^2_{\alpha,[2\hat{r}]}}\right)\right) = 1-\alpha$$

$$P(A_S \geq \hat{A}_{SL}(\alpha)) = 1-\alpha$$

where $[2\hat{r}]$ = smallest integer greater than or equal to $2\hat{r}$.

The procedure showed to be inaccurate due to small values of r.

D. PROCEDURE

The procedure used to fit a normal distribution to $1-\sum_{i=1}^{k}\hat{\alpha}_i\hat{\mu}_i$ as described in B is based on computer simulation.

The computations of the estimates of $\alpha_{i's}$ and $\beta_{i's}$ $(1=1, 2, \cdots k)$ are based on the assumption that we are given a series system with K types of components.

V_{ij} - time to failure of the j-th component of type i. The time to failure is assumed exponential with failure rate α_i.

T_{ij} - time to repair of the j-th component of type i. The time to repair is assumed exponential with repair rate β_i.

V_{ij} and T_{ij} are independent.

(1) A random sample of size N_i components of type i are tested until failure or a specified planned test time (PTT).

T_{oij} is the PTT of the j-th component of type i

$$X_{ij} = \begin{cases} 1 & \text{if the j-th component of type i fail, i.e., life time less than } T_{oij} \\ 0 & \text{if the j-th component of type i lives beyond } T_{oij}. \end{cases}$$

$$\sum_{j=1}^{N_i} X_{ij} = \text{\# of failures of type i components.}$$

The components were tested against the (PTT) T_{oij} and the observed operating times V_{oij} set equal to V_{ij} or T_{oij} whichever is the smaller.

(2) A random sample of size n_i of failed components of type i are repaired with repair times $T_{i1}, T_{i2}, \ldots, T_{in_i}$.

(3) Using the test data described in C 1 and 2 the estimates of α_i and μ_i were computed.

$$\hat{\alpha}_i = \frac{\sum_{j=1}^{N_i} X_{ij}}{\sum_{j=1}^{N_i} V_{oij}} \left(\frac{2 N_i}{2 N_i + 1} \right)$$

$$\hat{\mu}_i = \frac{\sum_{j=1}^{n_i} T_{ij}}{n_i}$$

(4) With the data in C-3 1000 values of $\hat{A}_{SL(\alpha)}$ were computed.

$$\hat{A}_{SL(\alpha)} = \hat{A}_S - \left(\sum_{i=1}^{K} \frac{\hat{\alpha}_i \hat{\mu}_i^2}{\sum_{j=1}^{N_i} T_{oij}} \right)^{\frac{1}{2}} Z_\alpha$$

where $\hat{A}_S = 1 - \sum_{i=1}^{K} \hat{\alpha}_i \hat{\mu}_i$ and

Z_{α} is the 100(1- α) th percentile point of the N(0, 1) distribution.

E. ANALYSIS

The generated values of $\hat{A}_{SL(\alpha)}$ was ordered from lowest to highest and the 1000(1- α)% one was compared to A_S.

The $\hat{A}_{SL(\alpha)}$ was also compared to A_S and the probability of success was computed by the formula

$$P(\hat{A}_{SL(\alpha)} \leq A_S)$$

based on the 1000 values of $\hat{A}_{SL(\alpha)}$.

The results are shown in Appendix A for the different combinations of α_i, β_i, N_i, n_i and T_{oij}. To avoid fluctuations due to the random number generator, the values of $\hat{A}_{SL(\alpha)}$ at different α's was computed by only changing Z_{α}.

III. ACCURACY RESULTS

By changing the values of the input parameters the accuracy of the procedure was checked. The following conclusions were made based on the simulation:

a. Use of the steady state availability of the system (A_S) to approximate the estimated availability $\hat{A}_{S\,L(\alpha)}$ over the mission time is a conservative procedure.

b. The amount by which the steady state availability differ from the estimated $100(1-\alpha)$th percentile point is very small.

Example

Case 3 - Appendix A.

With the given parameters the systems steady state availability is computed, giving

$$A_S = \prod_{i=1}^{k} \frac{\beta_i}{\beta_i + \alpha_i} = \underline{.9925}$$

The computed 95th percentile point of the distribution of $\hat{A}_{S\,L(0.05)}$ is equal to .9931

Thus, $\left| A_S - \hat{A}_{S(950)} \right| = \left| .9925 - .9931 \right| = \underline{.0006}$

Similarly the 90th and 80th percentile points of the distribution of $\hat{A}_{S\,L(.1)}$ and $\hat{A}_{S\,L(.2)}$ are .9930 and .9929 respectively. By increasing the number of items which are life tested from 50 to 100 and keeping the other parameters constant, we get the result given by case 5. That is,

$$A_S = .9925 \text{ (as before)}$$

$$\hat{A}_{S\,L(0.05)} = .9928$$

This implies that

$$\left| A_S - \hat{A}_{S(\ 950)} \right| = \left| .9925 - .9928 \right| = \underline{.0003}$$

In this case $\hat{A}_{S(\ 900)}$ and $\hat{A}_{S(\ 800)} = .9926$ which implies that

$$\left| A_S - \hat{A}_{S\,L(\alpha\ )} \right| = \underline{.0001}$$

for $\alpha = 0.1$ and $\alpha = 0.2$

The true levels of confidence associated with $\hat{A}_{S\,L(\alpha\ )}$ as a lower $100(1-\alpha\ )$ confidence limit for A_S are given in columns 10 through 12 for $\alpha = .05,\ .10,\ .20$ respectively. That is for Case 3, $\hat{A}_{S\,L(.05)}$ is an 90. 8% lower confidence limit for A_S rather than 95% lower confidence limit. This is a measure of the inaccuracy of $\hat{A}_{S\,L(.05)}$ as a lower 95% confidence limit for A_S for the given parameter values in Case 3. Likewise $\hat{A}_{S\,L(.10)}$ is really a 84. 5% lower confidence limit and $\hat{A}_{S\,L(.20)}$ is really a 76. 9% lower confidence limit.

This variation between the true level of confidence and the proposed level of confidence (e. g., 90. 8% vs. 95%) are due to the very small variance of $\hat{A}_{S\,L(\alpha\ )}$ for the cases considered. It is felt that for these cases the quantities

$$\left| A_S - \hat{A}_{S(1-\alpha\ )} \right|$$

are better measures for the accuracy of this procedure.

APPENDIX A

Result of the Simulation

Column	1	2	3	4	5	6	7	8	9	10	11	12
Case	α_i	β_i	N_i	n_i	T_{oij}	A_S	$A_{S(950)}$	$A_{S(900)}$	$A_{S(800)}$	$P(\hat{A}_{S(0.05)} \leqq A_S)$	$P(\hat{A}_{S(.1)} \leqq A_S)$	$P(\hat{A}_{S(.2)} \leqq A_S)$
1	.005	10.0	30	15	5.0	.9925	.9934	.9931	.9930	.9060	.8440	.7360
2	.005	10.0	30	15	2.0	.9925	.9953	.9942	.9937	.8560	.7970	.7060
3	.005	10.0	50	15	5.0	.9925	.9931	.9930	.9929	.9080	.8450	.7690
4	.005	10.0	50	15	2.0	.9925	.9944	.9937	.9934	.8660	.8120	.7320
5	.005	10.0	100	15	5.0	.9925	.9928	.9926	.9926	.9300	.8830	.7930
6	.005	10.0	100	15	2.0	.9925	.9934	.9930	.9928	.9070	.8570	.7540
7	.010	10.0	50	15	5.0	.9851	.9858	.9857	.9854	.9150	.8540	.7640
8	.010	10.0	50	15	2.0	.9851	.9864	.9861	.9858	.9040	.8470	.7600
9	.010	10.0	100	15	5.0	.9851	.9854	.9854	.9854	.9270	.8670	.7660
10	.010	10.0	100	15	2.0	.9851	.9859	.9853	.9851	.9270	.8840	.7970
11	.005	3.0	50	15	5.0	.9753	.9772	.9766	.9762	.9110	.8560	.7750
12	.005	3.0	50	15	2.0	.9753	.9814	.9790	.9779	.8720	.8150	.7400
13	.005	3.0	100	15	5.0	.9753	.9761	.9754	.9752	.9340	.8960	.8080
14	.005	3.0	100	15	2.0	.9753	.9780	.9767	.9761	.9080	.8610	.7680
15	.010	3.0	50	15	5.0	.9513	.9526	.9523	.9514	.9330	.8740	.7980
16	.010	3.0	50	15	2.0	.9513	.9548	.9538	.9526	.9090	.8620	.7730
17	.010	3.0	100	15	5.0	.9513	.9512	.9514	.9512	.9510	.8930	.8040
18	.010	3.0	100	15	2.0	.9513	.9529	.9511	.9505	.9370	.9920	.8230

```
      DIMENSION ALFA(15),BETA(15),V(15,100),X(15,100),T(15,60),
     1AHAT(15),YHAT(15),N(15),NI(15),AS1(1000),AS2(1000),AS3(1000)
      REAL PTT(15,100)/1500*2.0/
      MN=1000
      K=15
      Z05=1.645
      Z10=1.282
      Z20=0.842
      IX=934573
  100 FORMAT(2F10.5,2I10)
      DO 1000 I=1,K
      READ (5,100) ALFA(I),BETA(I),N(I),NI(I)
 1000 CONTINUE
      X1=0.0
      X2=0.0
      X3=0.0
      WRITE(6,500)
  500 FORMAT(///,33X,'ASHAT05',3X,'ASHAT10',3X,'ASHAT20',///)
  600 FORMAT(30X,3F10.4)
      AS=1.0
      DO 7000 I=1,K
 7000 AS=AS*BETA(I)/(ALFA(I)+BETA(I))
      DO 8000 M=1,MN
      DO 2000 I=1,K
      NN=N(I)
      DO 3000 J=1,NN
      X(I,J)=0.0
      CALL RANDU(IX,IY,YFL)
      IX=IY
      V(I,J)=-1.0/ALFA(I)*ALOG(YFL)
      IF(V(I,J).LT.PTT(I,J)) X(I,J)=1.0
      IF(V(I,J).GT.PTT(I,J)) V(I,J)=PTT(I,J)
 3000 CONTINUE
 2000 CONTINUE
      DO 3333 I=1,K
      NNI=NI(I)
      DO 4444 J=1,NNI
      CALL RANDU(IX,IY,YFL)
      IX=IY
      T(I,J)=-1.0/BETA(I)*ALOG(YFL)
 4444 CONTINUE
 3333 CONTINUE
      VA=0.0
      AYHAT=0.0
      DO 4000 I=1,K
      SUMT=0.0
      SUMXX=0.0
      SUMV=0.0
      NN=N(I)
      DO 5000 J=1,NN
      SUMXX=SUMXX+X(I,J)
      SUMV=SUMV+V(I,J)
      SUMT=SUMT+PTT(I,J)
 5000 CONTINUE
      SUMTT=0.0
      NNI=NI(I)
      DO 5555 J=1,NNI
      SUMTT=SUMTT+T(I,J)
 5555 CONTINUE
      XN=N(I)
      XNI=NI(I)
      AHAT(I)=SUMXX/SUMV*(2.0*XN)/(2.0*XN+1.0)
      YHAT(I)=SUMTT/XNI
      VA=VA+AHAT(I)*YHAT(I)**2/SUMT
      AYHAT=AYHAT+AHAT(I)*YHAT(I)
 4000 CONTINUE
      ASHAT=1.0-AYHAT
      AS1(M)=ASHAT-SQRT(VA)*Z05
      AS2(M)=ASHAT-SQRT(VA)*Z10
      AS3(M)=ASHAT-SQRT(VA)*Z20
      IF(AS.GE.AS1(M)) X1=X1+1.0
      IF(AS.GE.AS2(M)) X2=X2+1.0
      IF(AS.GE.AS3(M)) X3=X3+1.0
```

```
 8000 CONTINUE
      DO 8800 L=1,999
      JJ=L+1
      DO 8800 J=JJ,1000
      IF(AS1(L).LT.AS1(J)) GO TO 800
      TEMP1=AS1(L)
      AS1(L)=AS1(J)
      AS1(J)=TEMP1
  800 IF(AS2(L).LT.AS2(J)) GO TO 880
      TEMP2=AS2(L)
      AS2(L)=AS2(J)
      AS2(J)=TEMP2
  880 IF(AS3(L).LT.AS3(J)) GO TO 8800
      TEMP3=AS3(L)
      AS3(L)=AS3(J)
      AS3(J)=TEMP3
 8800 CONTINUE
      WRITE(6,600) (AS1(I),AS2(I),AS3(I),I=1,1000)
      WRITE(6,200)
  200 FORMAT(///,32X,'ALFA(I)',3X,'BETA(I)',7X,'N(I)',5X,'NN(I)',///)
      DO 9999 I=1,K
      WRITE(6,300) ALFA(I),BETA(I),N(I),NI(I)
 9999 CONTINUE
  300 FORMAT(30X,2F10.5,2I10)
      XMN=MN
      PROB1=X1/XMN
      PROB2=X2/XMN
      PROB3=X3/XMN
      WRITE(6,400) AS,PROB1,PROB2,PROB3,MN
  400 FORMAT(///,10X,'AS=',F7.4,4X,'PROB1=',F7.4,4X,'PROB2=',F7.4,4X,
     1'PROB3=',F7.4,4X,'MN=',I4)
      WRITE(6,700) AS1(950),AS2(900),AS3(800)
  700 FORMAT(///,10X,'AS1(950)=',F7.4,4X,'AS2(900)=',F7.4,4X,'AS3(800)='
     2,F7.4)
      STOP
      END
```

APPENDIX C

Derivation of Mean and Variance of $\hat{A}_S$

$$A_S = \prod_{i=1}^{K} A_i = \prod_{i=1}^{K} \frac{\beta_i}{\beta_i + \alpha_i} = \prod_{i=1}^{K} \frac{1}{1 + \frac{\alpha_i}{\beta_i}}$$

but $\frac{1}{\beta_i} = \mu_i$ = MTTR

The MTTR should always be less than one mission unit, which implies that

$$\alpha_i \mu_i < \alpha_i$$

thus

$$A_i = \frac{1}{1 + \alpha_i \mu_i} = 1 - \alpha_i \mu_i + (\alpha_i \mu_i)^2 - (\alpha_i \mu_i)^3 + \cdots\cdots$$

provided the series converges which it does since $\alpha_i \mu_i < 1$

Since the product $\alpha_i \mu_i$ is very small

$$A_i = \frac{1}{1 + \alpha_i \mu_i} \doteq 1 - \alpha_i \mu_i$$

Thus

$$A_S = \prod_{i=1}^{K} (1 - \alpha_i \mu_i) \doteq 1 - \sum_{i=1}^{K} \alpha_i \mu_i$$

Since $\hat{\alpha}_i$ and $\hat{\beta}_i$ will have small variance we shall apply the central limit theorem, and fit a normal distribution to

$$A_S = 1 - \sum_{i=1}^{K} \alpha_i \mu_i$$

Thus

$$\hat{A}_S = 1 - \sum_{i=1}^{K} \hat{\alpha}_i \hat{\mu}_i$$

when $\hat{\mu}_i$ is the estimated MTTR and $\hat{\alpha}_i$ is the estimated failure rate. Based on the assumption that time to failure and time to

repair is distributed exponentially we can find the mean and variance of $\hat{A}_S$.

$$E(\hat{A}_S) = 1 - \sum_{i=1}^{K} E(\hat{\alpha}_i) E(\hat{\mu}_i) = 1 - \sum_{i=1}^{K} \alpha_i \mu_i = A_S$$

By using the equations given by OD 29304 [3], for $\hat{\alpha}_i$ and $Var(\hat{\alpha}_i)$ we can find the variance for $\hat{A}_S$

$$\hat{\alpha}_i = \frac{\text{\# of failures}}{\text{\# of test time}} \cdot \frac{2N_i}{2N_i + 1}$$

$$E(\hat{\alpha}_i) = \alpha_i$$

$$Var(\hat{\alpha}_i) = \frac{\alpha_i}{\sum_j t_{ij}}$$

where $\sum_{j=1}^{N_i} t_{ij}$ = sum of all planned test times on component i (in mission units).

$$\hat{\mu}_i = \frac{\text{\# of repair times}}{n_i} = \frac{\sum_{j=1}^{n_i} T_{ij}}{n_i}$$

$$E(\hat{\mu}_i) = \mu_i$$

$$var(\hat{\mu}_i) = \frac{\mu_i^2}{n_i}$$

Thus
$$Var(\hat{A}_S) = Var\left(1 - \sum_{i=1}^{K} \hat{\alpha}_i \hat{\mu}_i\right)$$
$$= Var\left(\sum_{i=1}^{K} \hat{\alpha}_i \hat{\mu}_i\right)$$
$$= \sum_{i=1}^{K} Var(\hat{\alpha}_i \hat{\mu}_i) \qquad (1)$$

$$Var(\hat{\alpha}_i \hat{\mu}_i) = E(\hat{\alpha}_i^2 \hat{\mu}_i^2) - \alpha_i^2 \mu_i^2$$
$$= E(\hat{\alpha}_i^2) E(\hat{\mu}_i^2) - \alpha_i^2 \mu_i^2$$

But
$$E(\hat{\alpha}_i^2) = var(\hat{\alpha}_i) + \alpha_i^2$$

$$= \frac{\alpha_i}{\sum_j t_{ij}} + \alpha_i^2$$

$$E(\hat{\mu}_i^2) = Var(\hat{\mu}_i) + \mu_i^2$$

$$= \frac{\mu_i^2}{n_i} + \mu_i^2$$

$$= \mu_i^2 \left(\frac{n_i + 1}{n_i} \right)$$

Therefore $Var(\hat{\alpha}_i \hat{\mu}_i) = \left(\frac{\alpha_i}{\sum_j t_{ij}} + \alpha_i \right) \left(\mu_i^2 \cdot \frac{n_i + 1}{n_i} \right) - \alpha_i^2 \mu_i^2$

$$= \frac{\alpha_i \mu_i^2}{\sum_j t_{ij}} \cdot \frac{n_i + 1}{n_i} + \alpha_i^2 \mu_i^2 \cdot \frac{n_i + 1}{n_i} - \alpha_i^2 \mu_i^2$$

which implies that

$$var(\hat{\alpha}_i \hat{\mu}_i) \doteq \frac{\alpha_i \mu_i^2}{\sum_j t_{ij}} \cdot \frac{n_i + 1}{n_i}$$

Thus by (1)

$$var(\hat{A}_s) = \sum_{i=1}^{K} \frac{\alpha_i \mu_i^2}{\sum_{j=1}^{N_i} t_{ij}} \cdot \frac{n_i + 1}{n_i}$$

Since α_i and μ_i is approximately equal to $\hat{\alpha}_i$ and $\hat{\mu}_i$ respectively, and

$$\frac{n_i + 1}{n_i} \to 1 \text{ as } n_i \to \infty$$

we can write

$$E(\hat{A}_S) = 1 - \sum_{i=1}^{K} \hat{\alpha}_i \hat{\mu}_i \doteq A_S$$

$$var(\hat{A}_S) = \sum_{i=1}^{K} \frac{\hat{\alpha}_i \hat{\mu}_i^2}{\sum_{j=1}^{N_i} t_{ij}}$$

which implies that

$$\hat{A}_S \sim N(A_S, \ Var\ (A_S)) \doteq N(A_S, \ Var\ (\hat{A}_S))$$

$$= N\left(1 - \sum_i \hat{\alpha}_i \hat{\mu}_i, \ \sum_i \frac{\hat{\alpha}_i \hat{\mu}_i^2}{\sum_j t_{ij}}\right)$$

Thus by the above assumptions we can write

$$P\left(\frac{\hat{A}_S - A_S}{\sqrt{Var\ \hat{A}_S}} \leq Z_\alpha\right) = 1 - \alpha$$

$$P\left(\hat{A}_S - A_S \leq Z_\alpha \sqrt{Var(\hat{A}_S)}\right) = 1 - \alpha$$

$$P\left(-A_S \leq -\hat{A}_S + Z_\alpha \sqrt{Var\ \hat{A}_S}\right) = 1 - \alpha$$

$$P\left(A_S \geq \hat{A}_S - Z_\alpha \sqrt{Var\ \hat{A}_S}\right) = 1 - \alpha$$

Thus $\hat{A}_{SL(\alpha)} = \hat{A}_S - Z_\alpha \sqrt{Var\ (\hat{A}_S)}$

$$= \hat{A}_S - Z_\alpha \left(\sum_{i=1}^{K} \frac{\hat{\alpha}_i \hat{\mu}_i^2}{\sum_{j=1}^{N_i} t_{ij}}\right)^{\frac{1}{2}}$$

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13. ABSTRACT

The accuracy of proposed lower confidence limits for system availability is analyzed. Random values of the lower $100(1-\alpha)\%$ confidence limit $A_{SL(\alpha)}$ for system availability are computed for a system whose failure density is exponential (λ) and whose repair density is exponential (μ). The system is modeled as an alternating renewal process. The 100(1-) th percentile point of the generated distribution of $A_{SL(\alpha)}$ is compared with system availability as a measure of accuracy for $A_{SL(\alpha)}$.

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